Ordinary Differential Equations

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We have discussed antiderivatives of a given function $f(x)$, which can be obtained by computing the indefinite integral of $f(x)$, i.e., $F(x) = \int f(x)\,dx$. This is a good example of differential equations: recover the function $F(x)$ from some information about its derivatives or higher order derivatives.

More generally, an **ordinary differential equation** (ODE) is one of the form

$$G(x, y, y', ..., y^{(n)}) = 0$$

We assume $y = f(x)$ and solve for the expression of $f(x)$ from the above equation.

If the function involves more than one independent variables $x_1, ..., x_n$ instead of only an x, together with partial derivatives of y, and we solve for $y = f(x_1, ..., x_n)$, then the equation is called a **partial differential equation** (PDE).

ODE and PDE are two important subjects in modern mathematics.

Now we will deal with several types of ODEs.

Example 1. *Solve the differential equation*

$$y' = ry$$

where r is a constant, with initial condition $x = 0, y = 3$

$y' = ry$ implies $y' - ry = 0$. Multiplying on both sides by e^{-rx}, we see

$$y'e^{-rx} - re^{-rx}y = 0$$

by the chain rule, this is the same as

$$(ye^{-rx})' = 0$$

So there is constant C such that

$$ye^{-rx} = C$$

we get

$$y = \frac{C}{e^{-rx}} = Ce^{rx}$$

We now need to use the initial condition to determine C:

$$3 = Ce^{r\times 0} = C$$

So $C = 3$, the solution is

$$y = 3e^{rx}$$

Remark 2. There is another way to solve the above equation:

$y' = ry$ implies $\frac{y'}{y} = r$, by the chain rule this is nothing but $(\ln y)' = r$, so there is constant A such that

$$\ln y = rx + A$$

we get

$$y = e^{rx+A} = e^{A}e^{rx}$$

and the number e^{A} corresponds to the number C in the solution of the example.

Example 3. *Let $S(t)$ denote the sales volume of a particular commodity per unit of time, evaluated at time t . In a stable market where no sales promotion is carried out, the decrease in $S(t)$ per unit of time is proportional to $S(t)$. Thus sales decelerate at the constant proportional rate $a > 0$, implying that*

$$S'(t) = -aS(t)$$

(a) Find an expression for $S(t)$ when sales at time 0 are S_0.
(b) Solve the equation $S_0e^{-at} = \frac{1}{2}S_0$ for t . Interpret the answer.

The solution to this type of ODE are given by

$$S(t) = Ce^{-at}$$

and $S_0 = S(0) = Ce^{0} = C$, so we get

$$S(t) = S_0e^{-at}$$

Now solve

$$S(t) = S_0 e^{-at} = \frac{1}{2} S_0$$

we get $t = \frac{\ln 2}{a}$. *This is the time at which the sales volume reduces by half.*

Example 4. *Solve the differential equation:*

$$y' = a + by$$

where $b \neq 0$.

We can do a change of variable: let $u = a + by$, *then* $u' = by'$. *The equation becomes*

$$\frac{1}{b} u' = u$$

i.e.

$$u' = bu$$

So $u = Ce^{bx}$ *for some* C, *i.e.*, $a + by = Ce^{bx}$, *so the solutions are*

$$y = \frac{1}{b}(Ce^{bx} - a)$$

We can rewrite it as

$$y = Ae^{bx} - \frac{a}{b}$$

where A *is any constant*

There is a more general type of equation, called Separable Equations, which can be solved by a special method.

Definition 5. A separable equation is one of the form

$$y' = f(x)g(y)$$

For example, the ones we have discussed, $y' = ry$ and $y' = a + by$ are both separable equations.

Proposition 6. *A separable equation*

$$y' = f(x)g(y)$$

can be solved in the following steps:

1. *Rewrite the equation into the form*

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

2. *Integrate both sides:*

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

3. *Obtain y in terms of x from the above equation.*

Example 7. *The equation $y' = a + by$ can also be solved by viewing it as a separable equation:*

$$\begin{aligned} y' &= a + by \\ y' &= b\left(y + \frac{a}{b}\right) \\ \frac{1}{y + \frac{a}{b}}\,dy &= b\,dx \\ \int \frac{1}{y + \frac{a}{b}}\,dy &= \int b\,dx \\ \ln\left(y + \frac{a}{b}\right) &= bx + C \\ y + \frac{a}{b} &= e^{bx+C} \\ y &= -\frac{a}{b} + e^{bx}e^{C} \\ y &= Ae^{bx} - \frac{a}{b} \end{aligned}$$

Example 8. *We can use this method to solve*

$$y' = e^x y^2$$

We first write

$$\frac{1}{y^2}\,dy = e^x\,dx$$

Integrate both sides:

$$\int \frac{1}{y^2}\,dy = \int e^x\,dx + C$$

We get

$$-\frac{1}{y} = e^x + C$$

for some constant C. This implies

$$y = -\frac{1}{e^x + C}$$

Example 9. *Newtons Law of Cooling predicts the cooling of a warm body placed in a cold environment. According to the law, the rate at which the temperature of the body decreases is proportional to the difference of temperature between the body and its environment, i.e.*

$$\frac{dT}{dt} = k(T - T_e)$$

where T is the temperature of the object, T_e is the (constant) temperature of the environment, and k is a constant of proportionality. Given the initial temperature $T(0) = T_0$, we can recover the temperature function from this law:

$$\begin{aligned}
\frac{dT}{dt} &= k(T - T_e) \\
\frac{1}{T - T_e}\, dT &= -k\, dt \\
\ln(T - T_e) &= -kt + C \\
T &= T_e + e^{-kt+C}
\end{aligned}$$

When $t = 0, T = T_0$, so $T_0 = T_e + e^C$, we see $e^C = T_0 - T_e$, so

$$T = T_e + e^{-kt}(T_0 - T_e)$$

Observe that as $t \to +\infty$, $T \to T_e$, which means after long time, the temperature of the body will cool down to the environment temperature.

Example 10. *The Solow-Swan Model in economics is used to predict the long run growth in economics. It assumes at each time, the total production* *of some good (output) $Y(t)$ depends on the capital input $K(t)$ and labour input $L(t)$:*

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$

where $0 < \alpha < 1$ is the elasticity of the output with respect to capital input.

If We further assume the labour input is an exponential function of time given by

$$L(t) = e^{rt}$$

and the rate of increase of capital input is proportional to the total output at each time:

$$K'(t) = bY(t)$$

we can then solve for $K(t)$:

$$\begin{aligned}
\frac{1}{b}K' &= K^{\alpha}(e^{rt})^{1-\alpha} \\
\frac{1}{K^{\alpha}}\,dK &= be^{r(1-\alpha)t}\,dt \\
\int \frac{1}{K^{\alpha}}\,dK &= \int be^{r(1-\alpha)t}\,dt \\
\frac{1}{1-\alpha}K^{1-\alpha} &= \frac{b}{r(1-\alpha)}e^{r(1-\alpha)t} + C \\
K &= (\frac{b}{r}e^{r(1-\alpha)t} + C)^{\frac{1}{1-\alpha}}
\end{aligned}$$

Proposition 11. *The differential equation of the form*

$$y' + ay = f(x)$$

can be solved as follows:

1. *Multiply e^{ax} on both sides:*

   $$y'e^{ax} + ae^{ax}y = e^{ax}f(x)$$

   This is the same as

   $$(ye^{ax})' = e^{ax}f(x)$$

2. *the above can be written as*

   $$d(ye^{ax}) = e^{ax}f(x)\,dx$$

3. *Integrating both sides, we get*

$$ye^{ax} = \int e^{ax} f(x)\, dx + C$$

So $y = e^{-ax}(\int e^{ax} f(x)\, dx + C)$

Example 12. *Solve the differential equation*

$$y' + y = x$$

$$\begin{aligned}
y' + y &= x \\
y'e^x + ye^x &= xe^x \\
(ye^x)' &= xe^x \\
ye^x &= \int xe^x\, dx + C \\
ye^x &= xe^x - e^x + C \\
y &= x - 1 + Ce^{-x}
\end{aligned}$$